On average fault-tolerance in product graphs

E. Abajo, R. M. Casablanca, A. Diánez, and P. García-Vázquez
Department of Applied Mathematics I, University of Seville, Spain

EXTENDED ABSTRACT

Let $G$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The cardinalities
of these sets are denoted by $|V(G)| = n$ and $|E(G)| = e$. Let $u$ and $v$ be two distinct
vertices of $G$. A path from $u$ to $v$, also called an $uv$-path in $G$, is a subgraph $P$ with vertex
set $V(P) = \{u = x_0, x_1, \ldots, x_r = v\}$ and it is usually denoted by $P : x_0x_1\cdots x_r$. Two
$uv$-paths $P$ and $Q$ are said to be internally disjoint if $V(P) \cap V(Q) = \{u, v\}$. A cycle in
$G$ is a path $C : x_0x_1\cdots x_r$ such that $x_0 = x_r$. The girth of $G$, denoted by $g(G)$, is
the length of a shortest cycle in $G$, and if $G$ contains no cycles, then $g(G) = \infty$. The set of
adjacent vertices to $v \in V(G)$ is denoted by $N_G(v)$. The degree of $v$ is $d_G(v) = |N_G(v)|$,
whereas $\delta(G) = \min_{v \in V(G)} d_G(v)$ and $\overline{d}(G) = \frac{1}{n} \sum_{v \in V(G)} d_G(v) = 2e/n$ are the minimum
degree and the average degree of $G$, respectively. The connectivity $\kappa(G)$ of a graph $G$
is the smallest number of vertices whose deletion from $G$ produces a disconnected or a trivial
graph. Whitney [10] proved in 1932 that a graph $G$ is $r$-connected, that is, $\kappa(G) \geq r$, if and
only if every pair of vertices is connected by $r$ internally disjoint paths. From this result,
we know that the connectivity $\kappa_G(u, v)$ between two distinct vertices $u$ and $v$ in $G$ is the
maximum number of pairwise internally disjoint $uv$-paths in $G$. In this way, the connectivity
of a graph can be seen as $\kappa(G) = \min_{u, v \in V(G)} \kappa_G(u, v)$. In [10] the author also showed that
$\kappa_G \leq \delta(G)$. The graph $G$ is maximally connected if the previous bound is attained, that
is, if $\kappa(G) = \delta(G)$.

For a graph $G$ of order $n$, the average connectivity $\overline{\kappa}(G)$ is defined as the average of the
connectivities between all pairs of vertices of $G$, that is,

$$\overline{\kappa}(G) = \frac{1}{\binom{n}{2}} \sum_{u, v \in V(G)} \kappa_G(u, v).$$

In order to avoid fractions, we also consider the total connectivity $K(G)$ of $G$, defined
as $K(G) = \sum_{u, v \in V(G)} \kappa_G(u, v)$. While the connectivity is the minimum number of vertices
whose removal separates at least one connected pair of vertices, the average connectivity is a
measure for the expected number of vertices that have to be removed to separate a randomly
chosen pair of vertices.

It is well known that most networks can be modeled by a graph $G = (V, E)$. The best
known measure of reliability of a graph is its connectivity, defined above. As the connectivity
is a worst-case measure, it does not always reflect what happens throughout the graph. For
example, a tree and the graph obtained by appending an end-vertex to a complete graph both
have connectivity 1. Nevertheless, for large order the latter graph is far more reliable than
the former. Interest in the vulnerability and reliability of networks such as transportation
and communication networks, has given rise to a host of other measures of reliability, see
for example [1]. In this paper we pay attention to a measure for the reliability of a graph, the
average connectivity, introduced by Beineke, Oellermann and Pippert [3].

There is a lot of research on the connectivity of a graph (see [8]). Many works provide
suﬃcient conditions for a graph to be maximally connected or super connected [4]. Others
study the maximal connectivity in networks that are constructed from graph generators, as
cartesian product graphs [5, 9] or permutation graphs [2, 7].

There are two excellent papers where the average connectivity has been investigated.
In the first one, Beineke, Oellermann and Pippert [3] find upper and lower bounds on the
average connectivity of a graph $G$ in terms of its order $n$ and its average degree $\overline{d}(G)$. In the
second one, Dankelmann and Oellermann [6] obtain sharp upper bounds for some families
of graphs. We study the average connectivity of the so-called strong product of graphs.
For a large system, configuration processing is one of the most tedious and time-consuming parts of the analysis. Different methods have been proposed for configuration processing and data generation. Some of them are structural models which can be seen as the product graph of two given graphs, known as generators. Many properties of structural models can be obtained by considering the properties of their generators. In this sense, a usual objective in network design is the extension of a given interconnection system to a larger and fault-tolerant one so that the communication delay among nodes of the new network is small enough. To achieve this goal, many works in Graph Theory have studied fault-tolerant properties of some products of graphs.

We focus on the strong product $G_1 \boxtimes G_2$ of two graphs $G_1$ and $G_2$ is defined on the cartesian product of the vertex sets of the generators, so that two distinct vertices $(x_1, x_2)$ and $(y_1, y_2)$ of $G_1 \boxtimes G_2$ are adjacent if $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$, or $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_1 y_1 \in E(G_1)$ and $x_2 y_2 \in E(G_2)$.

In this work we provide, by a constructive method, a lower bound on the average connectivity of the strong product $G_1 \boxtimes G_2$ of two connected graphs $G_1$ and $G_2$ of girth at least 5. As a consequence, we prove that the strong product of two maximally connected graphs of girth at least 5 is maximally connected, and also, that $\kappa(G_1 \boxtimes G_2) = \delta(G_1 \boxtimes G_2)$ if $\kappa(G_i) = \delta(G_i)$, $i = 1, 2$.

Acknowledgment

This research was supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2011-28800-C02-02; and under the Catalonian Government project 1298 SGR2009.

References


